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Multiobjective optimization of credit capital allocation in financial institutions

Kamil J. Mizgier · Joseph M. Pasia

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Abstract The evolution of international regulation leads to new capital requirements imposed on globally active companies. Financial services firms allocate capital to business lines in order to withstand the materializing credit losses and to measure the performance of various business lines. In this study, we introduce a methodology for optimal credit capital allocation based on operations research approach. In particular, we focus on the efficient allocation of capital to business lines characterized by credit risk losses and cost of capital. We compare different allocation methods and provide a rationale behind using the OR approach. Finally, we formulate a multiobjective optimization model to capital allocation problem and apply it to a real-world case of two financial conglomerates.

Keywords Risk management · Capital allocation · Multiobjective optimization · Cost of capital · OR in banking

1 Introduction

With the introduction of Basel III, globally active financial institutions are required to strengthen their loss absorbing capital with the aim to decrease the systemic risk of the entire banking sector. This international standard has been coined by the Basel Com-

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K. J. Mizgier (✉)
Chair of Logistics Management, Department of Management, Technology, and Economics,
Swiss Federal Institute of Technology Zurich, Weinbergstrasse 56/58, 8092 Zurich, Switzerland
e-mail: kmizgier@ethz.ch

J. M. Pasia
UBS AG, Freischuetzgasse 25, 8004 Zurich, Switzerland

mittee on Banking Supervision (BCBS). As the 2008–2012 financial crisis showed, banks were not adequately capitalized to deal with the cascading effects of the global financial meltdown. The focus of regulators has shifted towards increasing the banks' capital levels and introducing more transparency into the Tier 1 capital calculations (BCBSa 2011). Moreover, the postulates of the reduction of procyclicality and promoting countercyclical buffers introduce new requirements on the usage of available capital. The regulators are actively designing stringent capital rules for financial conglomerates and have taken steps towards the standardization of the economic capital calculations (BCBSb 2009). The policy enables banks to choose their internal method for calculating risk in terms of economic capital, which results in the flexibility of choice, but also in the difficulties in implementation of "best practices." A combination of the available Tier 1 capital (risk capacity) and the economic capital (risk exposure) calculations naturally leads to a discussion on the sound methods for optimal allocation of capital to the various risk types and business lines (BLs). On the one hand, the currently used methods for capital allocation are often not sufficiently sophisticated to capture all dependencies across the risk types and the changing complexity of today's businesses. On the other hand, the literature on the efficient and optimal capital allocation in practice is scarce. Banks need to optimize capital allocation not only to conform to international regulatory guidelines, but also to manage the capital consumption related to certain business activities. In particular, it is important to decouple the institution-wide capital allocation and the capital allocation to different BLs, as each of them represents different business and is characterized by a BL-specific cost of capital. Naturally, this topic becomes one of high importance both from the researchers' and practitioners' perspectives as it is relevant for risk managers, policy makers and regulators around the globe. However, due to the lack of industry standards and various challenges with technical implementation, no "best practice" exists up to date.

Optimal capital allocation belongs to a wide class of decision problems under uncertainty that in finance, supply chain management, project and portfolio management or energy production are solved by stochastic programming methods. These methods are well established in operations research and include linear and nonlinear stochastic optimization problems, with large numbers of scenarios and dynamic stochastic optimization. In this article, we propose that combining methodologies from finance and operations research is a promising approach to the design of a new international industry standard. The aim of our research is to extend the view of the optimal capital allocation problems to a more general class of multiobjective resource allocation problems.

The novelty of our study lies in the consideration of the cost of credit capital allocated to the BLs. This leads to a bi-objective formulation of the capital allocation problem which may lead to conflicting solutions. Furthermore, we provide analytical solutions of the allocation problem by using both the geometric arguments and by applying Lagrange multipliers. Finally, as opposed to other (purely theoretical) studies, we test our model with real-world data in order to make our method more accessible to practitioners.

The remainder of this article is organized as follows. In Sect. 2 we present the relevant literature. In Sect. 3 we provide an overview of the existing methods for optimal capital allocation. In Sect. 4 we elaborate on the analytical properties of the model. Then, in Sect. 5 we describe the data used in simulations, we analyze the model specifications and discuss the results. Section 6 concludes and suggests future research paths.

2 Literature review

In the traditional finance literature, capital allocation problems are mainly studied in applications to insurance and banking industries. While insurance-related research provides a thorough theoretical underpinning of the underlying concepts, research on banking has focused more on the applications in practice and on the development of international regulatory guidelines. A large body of literature focuses on the proportional allocation of capital to the corresponding BLs (Lin et al. 2013). The main application areas of the capital allocation methods are in risk measurement, risk appetite estimation and performance-based capital allocation. The first line of research is well represented by the work of Cummins (2000) where the author provides an overview of the various techniques for allocating equity capital to several BLs of the insurance companies. In terms of coherent risk measures Denault (2001) and Kalkbrener (2005) propose an axiomatic system for capital allocation and analyze popular risk measures in the financial industry. A breakthrough paper for calculating capital allocation in the actuarial science is the work of Myers and Read (2001). Tasche (2004) provides an economically meaningful method of internal capital allocation based on Euler principles. The gradient allocation principle is further studied in the context of coherent risk measures and coherent capital allocation by Buch and Dorfleitner (2008). Mausser and Rosen (2007) provide an overview of the measurement methods of economic credit capital and their application to capital allocation. A unifying approach to the optimal capital allocation methods is presented by Dhaene et al. (2012). The article provides an extensive discussion of the existing methods for capital allocation and proposes a new approach formulated as an optimum allocation problem. Several authors have recently suggested variations of the optimum allocation problem. For instance, Xu and Hu (2012) study capital allocations via stochastic comparisons in the general loss function scenarios. In another article, Xu and Mao (2013) introduce a new capital allocation rule based on the tail mean-variance principle.

In operations research, optimum allocation problems have been studied for a long time and cover a wide array of topics ranging from finance, logistics, production economics, search theory to inventory models (Steuer and Na 2003; Patriksson 2008). For instance, the optimal configuration of a production and distribution network subject to operational and financial constraints has been studied by Basso and Peccati (2001). A mixed integer linear programming (MILP) model is proposed to describe the optimization problem. By using the MILP algorithm (Tsiakis and Papageorgiou 2008) solve optimal production allocation problems in supply chain networks. Financial constraints include production costs, transportation costs and duties for the material flowing within the network subject to exchange rates. Another area of optimal allocation problems that is well represented in the literature copes with the portfolio optimization methods (Kolm et al. 2014). For instance, Li et al. (2013) propose a model for allocating the systematic risk in portfolio optimization, while Fonseca and Rustem (2012) develop a robust optimization method for international portfolio management and introduce a tractable semidefinite programming formulation of the model. Homburg and Scherpereel (2008) present a novel approach to allocate the cost of risk capital in decentralized organizations. The authors elaborate on fair risk capital allocation schemes from a behavioral perspective. In terms of the dynamics of optimal capital allocation, Estrella (2004) formulates the infinite-horizon stochastic optimization problem in presence of costs. An optimization model of quality improvement allocations that minimize costs is studied by Wang et al. (2013).

When both literature streams are viewed collectively, it appears that the application of operations research approach to the capital allocation theory has not been given the attention that it deserves. In our research, we try to bridge this research gap by formulating a multiobjective capital allocation problem that draws on the theoretical foundations of finance and applied operations research literature.

3 Principles and methods of capital allocation problem

Consider a financial conglomerate with n different lines of business (which can include several international locations) with losses represented by random variables $L_1, L_2, \ldots, L_n$ occurring at a fixed future date T. We assume that any loss L_i has a finite mean i.e., $E[L_i] = \mu_i < +\infty$, $i = 1, 2, \ldots, n$ and denote its distribution function $P(L_i \leq l)$ by $F_{L_i}(l)$.

Given the aggregate loss $L = \sum_{i=1}^{n} L_i$ and a risk measure ρ such as the value-at-risk (VaR) or expected shortfall (ES), we assume that the conglomerate has already determined the overall risk capital K given by $K = \rho(L) > 0$. The principle of capital allocation can then be described as allocating the risk capital K to various BLs such that, if K_i is the amount of capital allocated to the BL with potential loss L_i, then it satisfies the full allocation requirement given by

$$\sum_{i=1}^{n} K_i = K. \tag{1}$$

Erel et al. (2014) suggest that risk-free or low-risk businesses may get negative capital allocations, since their marginal expansion improves the conglomerate's credit quality which frees up the risk capital.

There are many methods for obtaining the allocated risk capitals $K_1, K_2, \ldots, K_n$. One class of methods is based on the proportional allocation principle (Dhaene et al. 2003) which states that each capital K_i is proportional to the risk measure of its associated loss L_i i.e.,

$$K_i = \gamma_i \rho(L_i), \quad i = 1, 2, \ldots, n. \tag{2}$$

The definition of the factor γ_i differentiates several methods of this class. One popular approach is to define γ_i as

$$\gamma_i = \frac{K}{\sum_{j=1}^{n} \rho(L_j)}, \quad i = 1, 2, \ldots, n, \tag{3}$$

which clearly satisfies the allocation requirement in Eq. 1.

Another common method is the covariance allocation principle (Litterman 1996; Overbeck 2000) which allocates the risk capital as follows

$$K_i = \frac{K}{\mathrm{Var}(L)}\mathrm{Cov}(L_i, L), \quad i = 1, 2, \ldots, n, \tag{4}$$

where $\mathrm{Var}(L)$ is the variance of the aggregate loss and $\mathrm{Cov}(L_i, L)$ is the covariance between the individual loss L_i and aggregate loss L. Clearly, this method also satisfies the full allocation requirement since the sum of the covariances is equal to the variance of the aggregate loss. Unlike in the proportional allocation principle, the use of covariance allows the method to explicitly model the dependence structure of the random losses where individual losses that are highly correlated with the aggregate loss have higher allocated capital than those losses with lower correlations (for a detailed discussion, please see McNeil et al. 2010).

The conditional tail expectation (CTE) allocation principle (Tasche 1999; Acerbi and Tasche 2002) is another alternative method that also includes the dependence structure of the random losses. As the name implies, it uses the CTE as risk measure for setting the capital requirements. For a fixed probability level $p \in (0, 1)$, it allocates a capital K_i given by

$$K_i = \frac{K}{\mathrm{CTE}_p[L]}\mathbb{E}\left[L_i | L > F_L^{-1}(p)\right], \quad i = 1, 2, \ldots, n, \tag{5}$$

where F_L^{-1} is the quantile function and the CTE of the aggregate loss L is defined as the average of the top $1 - p$ percent losses i.e.,

$$\mathrm{CTE}_p[L] = \mathbb{E}\left[L | L > F_L^{-1}(p)\right]. \tag{6}$$

The Euler or (also termed) gradient allocation principle is another method that calculates the per-unit allocation based on the gradient of a positive-homogeneous risk measure (Buch and Dorfleitner 2008). Using appropriate risk measures, this method can obtain many allocation principles including the covariance, semicovariance and Expected Shortfall allocation principles. Buch and Dorfleitner (2008) investigate various conditions under which these three principles are non-coherent in the general context of risk measures (Artzner et al. 1999) and capital allocations (Denault 2001).

In Laeven and Goovaerts (2004), a capital allocation approach that integrates the cost of economic capital and the cost of positive risk is proposed. The positive risk, also called risk residual, is the risk that remains after allocating the capital. This model first obtains the optimal total economic capital X^* via a mathematical model given by

$$\min_X \pi\left[(L - X)_+\right] + (r_c - r_f)X, \tag{7}$$

where X denotes the solution vector $(K_1, K_2, \ldots, K_n)$, r_c is the cost of capital (or cost of capital charged by the shareholders), r_f $(\le r_c)$ is the risk-free interest rate, and $\pi[\cdot]$ is a valuation measure for the risk residual. Note that the first term of Eq. 7

reflects the risk residual while the second term represents the cost of capital allocation. Next, the cost of economic capital due to each of the BLs is similarly calculated using Eq. 7. Finally, each of the various BLs is then allocated an amount equal to the fraction $\pi[(L-X^*)_+]/\sum_{j=1}^{n}\pi[(L_j-X_j^*)_+]$ of the risk residual cost of the BL $\pi[(L_i-X_i^*)_+]$ plus the capital cost $(r_c-r_f)X_i^*$.

Most recently, a unifying framework that generalizes several allocation approaches has been developed in Dhaene et al. (2012). This framework is a mathematical programming method based on the minimization of the deviations of the BLs' losses from their respective allocated capitals. It is characterized by its flexibility which allows the derivations, interpretations and extensions of other existing methods. This framework can be seen as a general form of the allocation models presented in Laeven and Goovaerts (2004).

Mathematically, the framework of optimal allocation problem determines the allocated capitals $K_1, K_2, \ldots, K_n$ by solving the following optimization model

$$\min_{K_1,K_2,\ldots,K_n} \sum_{i=1}^{n} v_i \mathbb{E}\left[e_i D\left(\frac{L_i-K_i}{v_i}\right)\right] \text{ subject to } \sum_{i=1}^{n} K_i = K, \tag{8}$$

where the v_i and e_i are non-negative real numbers and non-negative random variables, respectively, such that $\sum_{i=1}^{n} v_i = 1$ and $\mathbb{E}[e_i] = 1$, and D is a non-negative function.

The parameters v_i normalize the deviations of losses from allocated capitals in the function D to make the deviations more comparable across BLs. Moreover, they are also used as weights to indicate the relative importance of the different BLs. Likewise, the parameters e_i are used as weights for the function D. For example, one can define e_i as a non-negative and non-decreasing function of loss L_i, thus assigning larger weights to deviations that correspond to the larger losses L_i. Several possible forms of e_i are described in Dhaene et al. (2012).

The function D evaluates the difference between the losses L_i and the allocated capitals K_i. One example of this function is the quadratic deviation given by $D(\cdot) = \frac{(L_i-K_i)^2}{v_i^2}$. It can be shown that the optimal allocated capitals under the quadratic deviation are

$$K_i^* = \mathbb{E}[e_i L_i] + v_i \left(K - \sum_{j=1}^{n} \mathbb{E}[e_j L_j]\right) \quad i = 1, 2, \ldots, n. \tag{9}$$

If one defines the parameter v_i by

$$v_i = \frac{\mathbb{E}[e_i L_i]}{\sum_{j=1}^{n} \mathbb{E}[e_j L_j]}, \quad i = 1, 2, \ldots, n, \tag{10}$$

then the optimal allocated capitals can be expressed as

$$K_i = K \frac{\mathbb{E}[e_i L_i]}{\sum_{j=1}^{n} \mathbb{E}[e_j L_j]}, \quad i = 1, 2, \ldots, n, \tag{11}$$

which follows the proportional allocation principle.

4 Multiobjective model

Our multiobjective optimization model to capital allocation problem (MOMCAP) combines the cost of allocation, which is derived from the risk residual defined in Laeven and Goovaerts (2004), to the general framework of capital allocation optimization model described in Dhaene et al. (2012). The former study also discusses the economic motivation for including costs in the optimal capital allocation problem. Our model also closely resembles the lot sizing problem from the resource allocation models studied in production economics (Patriksson 2008). In particular, MOMCAP aims to simultaneously minimize the total cost of allocation C and the squared deviation D between allocated capitals and losses. We define the cost of allocation of capital c_i as the cost per unit amount of capital allocated to BL i. The total cost of allocation of capital is then given by

$$C = \sum_{i=1}^{n} c_i K_i. \tag{12}$$

To fully describe our model, we denote as X the solution vector $(K_1, K_2, \ldots, K_n)$. Since both C and D are functions of the solution vector X, we can express them as $C(X)$ and $D(X)$. We also abbreviate as S the set of all feasible solutions. Finally, the general form of MOMCAP can be described as follows

$$\min\,(f_1(X), f_2(X)) \quad \text{such that } X \in S, \tag{13}$$

where $f_1(X) = D(X)$ and $f_2(X) = C(X)$.

It can easily be verified that the two objectives of MOMCAP are conflicting. For instance, the optimal solution of squared deviation D is given by Eq. 9 while the optimal solution for cost of allocation C is

$$K_i^* = K, \quad \text{where } i = \text{argmin}(c_1, c_2, \ldots, c_n). \tag{14}$$

Clearly, the optimal solution in one objective does not translate to an optimal solution in the other objective. In general, solving multiobjective problems like MOMCAP means finding the set of nondominated solutions or the so-called Pareto-optimal set.[1]

For our MOMCAP model, we seek the Pareto optimal set by solving the optimization problem (13) via an aggregation approach i.e., we solve the corresponding single-objective function given by

[1] A solution $Z \in S$ is said to dominate a solution $Y \in S$ if $f_i(Z) \leq f_i(Y)$ for all $i = 1, 2, \ldots, m$ and $f_j(Z) < f_j(Y)$ for at least one index $1 \leq j \leq m$. Moreover, Z is said to be not dominated by set S if there is no solution $Y \in S$ that dominates Z. In this case, Z is called nondominated or Pareto-optimal solution. The collection of all Pareto-optimal solutions is called the Pareto-optimal set and the graphical representation of these solutions in terms of their objective function values is called the efficient frontier.

$$\begin{cases} \min\ [(1-\alpha)\times f_1(X)+(\alpha)\times f_2(X)] \\ \text{subject to } \sum_{j=1}^{n} K_j = K \text{ and } \alpha = [0,1]. \end{cases} \tag{15}$$

By varying the values of α, we can generate the efficient frontier of our model. We obtain the analytical derivation of the Pareto optimal solutions by writing our optimization problem as follows:

$$\begin{cases} \min(f_1, f_2) = (1-\alpha)\sum_{j=1}^{n} v_j \mathbb{E}\left[e_j \frac{(K_j - L_j)^2}{v_j^2}\right] + (\alpha)\sum_{j=1}^{n} c_j K_j \\ \text{subject to } \sum_{j=1}^{n} K_j = K \quad \text{and} \quad \alpha = [0,1]. \end{cases} \tag{16}$$

4.1 Analytical solution based on geometric arguments

In what follows, we present an analytical solution to the MOMCAP problem. Again, the optimal solutions when $\alpha = 0$ and $\alpha = 1$ are already given in Eqs. 9 and 14, respectively. Following Dhaene et al. (2012), Zaks (2013) and Zaks and Tsanakas (2014) our solution when $0 < \alpha < 1$ is based on the transformation of the objective function to a form where only decision variables are left and we use geometric arguments to obtain the final solution. We set

$$r_j = \left(\frac{1-\alpha}{v_j}\right)^{1/2} \tag{17}$$

and begin by transforming f_1,

$$f_1 = (1-\alpha)\sum_{j=1}^{n} v_j \mathbb{E}\left[e_j \frac{(K_j - L_j)^2}{v_j^2}\right] \tag{18}$$

$$= (1-\alpha)\sum_{j=1}^{n} \frac{1}{v_j}\mathbb{E}\left[e_j K_j^2 - 2e_j K_j L_j + e_j L_j^2\right] \tag{19}$$

$$= \sum_{j=1}^{n} (r_j K_j)^2 - 2r_j^2 K_j \mathbb{E}(e_j L_j) + r_j^2 \mathbb{E}\left(e_j L_j^2\right). \tag{20}$$

It can be rewritten as:

$$f_1 = \sum_{j=1}^{n} (r_j K_j)^2 - 2r_j^2 K_j \mathbb{E}(e_j L_j) + r_j^2 \mathbb{E}\left(e_j L_j^2\right) + r_j^2 \mathbb{E}\left(e_j^2 L_j^2\right) - r_j^2 \mathbb{E}\left(e_j^2 L_j^2\right) \tag{21}$$

$$= \sum_{j=1}^{n} \left[r_j K_j - r_j \mathbb{E}(e_j L_j)\right]^2 + r_j^2 \mathbb{E}(e_j L_j^2) - r_j^2 \mathbb{E}\left(e_j^2 L_j^2\right). \tag{22}$$

Since the last two terms in the equation above do not include the decision variable K_j then dropping them will give the same optimal solution, thus f_1 can be expressed as:

$$f_1 = \sum_{j=1}^{n} \left[r_j K_j - r_j \mathbb{E}(e_j L_j) \right]^2 . \tag{23}$$

Moreover, if we let $s_j = \mathbb{E}\left[e_j L_j\right] r_j$ and $d_j = \alpha c_j$ then a simple transformation leads to the following formulation of MOMCAP:

$$\min_{K_1, K_2, \ldots, K_n} \left[\sum_{j=1}^{n} \left(r_j K_j - s_j\right)^2 + \sum_{j=1}^{n} d_j K_j \right]. \tag{24}$$

Note that

$$\sum_{j=1}^{n} \left(r_j K_j - s_j\right)^2 + \sum_{j=1}^{n} d_j K_j = \sum_{j=1}^{n} \left[\left(r_j K_j\right)^2 - 2 r_j K_j s_j + s_j^2 + d_j K_j \right]. \tag{25}$$

By using basic algebraic transformations and removing terms independent of decision variables $K_1, K_2, \ldots, K_n$, we obtain the following form of the problem:

$$\min_{K_1, K_2, \ldots, K_n} \left[\sum_{j=1}^{n} \left(r_j K_j - \left(s_j - \frac{d_j}{2 r_j} \right) \right)^2 \right] = \min_{K_1, K_2, \ldots, K_n} \left[\sum_{j=1}^{n} \left(r_j K_j - T_j \right)^2 \right], \tag{26}$$

where $T_j = s_j - \frac{d_j}{2 r_j}$. Let $U_j = r_j K_j$ and $R_j = \frac{1}{r_j}$. We introduce a vector notation of MOMCAP, where $\mathbf{U} = (U_1, \ldots, U_n)$, $\mathbf{T} = (T_1, \ldots, T_n)$ and $\mathbf{R} = \left(\frac{1}{r_1}, \ldots, \frac{1}{r_n}\right)$. We rewrite Eq. 26 as:

$$\begin{cases} \min\limits_{U,T} \|\mathbf{U} - \mathbf{T}\|^2 \\ \text{subject to } \langle \mathbf{U}, \mathbf{R} \rangle = K. \end{cases} \tag{27}$$

The geometric solution to the optimization problem (27), which is similar to finding the point on the hyperplane defined by the constraint $\langle \mathbf{U}, \mathbf{R} \rangle = K$ closest to the point $\mathbf{T}$, is given by:

$$\mathbf{U}^* = \frac{K - \langle \mathbf{T}, \mathbf{R} \rangle}{\langle \mathbf{R}, \mathbf{R} \rangle} \mathbf{R} + \mathbf{T}. \tag{28}$$

Solving for K_i, we obtain the optimal solution

$$K_i^* = \frac{1}{r_i} \left[\frac{K - \langle \mathbf{T}, \mathbf{R} \rangle}{\langle \mathbf{R}, \mathbf{R} \rangle} R_i + T_i \right]. \tag{29}$$

After expanding **T** and **R**, the optimal solution of MOMCAP is given by:

$$K_i^* = \mathbb{E}(e_i L_i) + v_i \left[K - \sum_{j=1}^{n} \mathbb{E}(e_j L_j) \right] + \frac{v_i \alpha}{2(1-\alpha)} \left[\sum_{j=1}^{n} v_j c_j - c_i \right]. \tag{30}$$

Note that for special case $\alpha = 0$, $K_i^* = \mathbb{E}(e_i L_i) + v_i \left[K - \sum_{j=1}^{n} \mathbb{E}(e_j L_j) \right]$ which is equal to Eq. 9. Moreover, the last term in Eq. 30 expresses the difference between the average cost of allocation $\sum_{j=1}^{n} v_j c_j$ and the cost of allocation c_i. Thus, BLs gain/loss additional amount of allocated capital when their cost of allocation is lower/higher than the average.

4.2 Analytical solution based on Lagrange multipliers

Alternatively, we propose a solution based on Lagrange multipliers. We define the Lagrange function as:

$$L\left(\bar{K}, \lambda\right) = (1-\alpha) \sum_{j=1}^{n} v_j \mathbb{E}\left[e_j \frac{\left(K_j - L_j\right)^2}{v_j^2} \right] + \alpha \sum_{j=1}^{n} c_j K_j - \lambda \left(\sum_{j=1}^{n} K_j - K \right), \tag{31}$$

which yields the following system of equations

$$\frac{\partial L}{\partial K_i} = \frac{2(1-\alpha)}{v_i} \mathbb{E}\left(e_i K_i - e_i \bar{L}_i\right) + \alpha c_i - \lambda = 0 \tag{32}$$

$$\frac{\partial L}{\partial \lambda_i} = -\sum_{j=1}^{n} K_j + K = 0. \tag{33}$$

They can be rewritten as

$$\frac{2(1-\alpha)}{v_i} \left[K_i - \mathbb{E}\left(e_i \bar{L}_i\right) \right] + \alpha c_i - \lambda = 0 \tag{34}$$

$$\sum_{j=1}^{n} K_j = K. \tag{35}$$

Equation 34 yields

$$K_i = \frac{v_i}{2(1-\alpha)} \left[\lambda - \alpha c_i \right] + \mathbb{E}\left(e_i \bar{L}_i\right). \tag{36}$$

From Eqs. 35 and 36 we get

$$\sum_{j=1}^{n} K_j = K = \sum_{j=1}^{n} \frac{v_j}{2(1-\alpha)} \left[\lambda - \alpha c_j \right] + \sum_{j=1}^{n} \mathbb{E}\left(e_j \bar{L}_j\right). \tag{37}$$

First, we solve for λ

$$K = \sum_{j=1}^{n} \frac{v_j}{2(1-\alpha)} \left[\lambda - \alpha c_j\right] + \sum_{j=1}^{n} \mathbb{E}\left(e_j \bar{L}_j\right). \tag{38}$$

Note that $\sum_{j=1}^{n} v_j = 1$, which yields

$$\lambda = 2(1-\alpha) \left[\left(K - \sum_{j=1}^{n} \mathbb{E}\left(e_j \bar{L}_j\right)\right)\right] + \alpha \sum_{j=1}^{n} v_j c_j. \tag{39}$$

Now, we solve for K_i. From Eqs. 36 and 39 we obtain the optimal solution:

$$K_i^* = \mathbb{E}(e_i L_i) + v_i \left[K - \sum_{j=1}^{n} \mathbb{E}\left(e_j L_j\right)\right] + \frac{v_i \alpha}{2(1-\alpha)} \sum_{j=1}^{n} v_j c_j - \frac{v_i \alpha}{2(1-\alpha)} c_i. \tag{40}$$

Finally, the optimal solution can be rewritten as:

$$K_i^* = \mathbb{E}(e_i L_i) + v_i \left[K - \sum_{j=1}^{n} \mathbb{E}(e_j L_j)\right] + \frac{v_i \alpha}{2(1-\alpha)} \left[\sum_{j=1}^{n} v_j c_j - c_i\right], \tag{41}$$

which is identical to Eq. 30.

5 Results of numerical simulations

5.1 Data

In March 2013 the Federal Reserve Board published the results of the Dodd–Frank Act Stress Test 2013: Supervisory Stress Test Methodology and Results (FED 2013). The Federal Reserve expects large, complex bank holding companies (BHCs) to hold sufficient capital to continue lending to support real economic activity, even under adverse economic conditions. In total 18 BHCs have been asked to provide input data through a series of forms. The report provides an overview of the analytical framework and methods used to generate the projections of loan losses for each of the 18 BHCs. The Federal Reserve's projections depict possible results under hypothetical, severely adverse conditions. For this study, we selected two sufficiently complex and heterogeneous financial conglomerates to allow a meaningful comparative analysis. We chose American Express (AE) and Bank of America (BofA) as representative firms for our analysis. We parametrized our simulation setup with the corresponding loss data as in the FED report by BL. The following BLs are captured by the FED report: first-lien mortgages, junior liens and HELOCs, commercial and industrial, commercial real estate, credit cards, other consumer and other loans. We assumed that the stress figures correspond to the credit losses in the stressed scenarios, and that the values reported in the FED report are the mean values of a Gaussian distribution. Naturally, a different than normal distribution of losses around the mean value would lead to different results, however, only for the purpose of our presentation the assumption is justified. In Table 1 we present the list of parameters.

Note that BofA (BL1–BL7) has more BLs than AE (BL3, BL5, BL7) which expresses the characteristics and complexity of their respective business models. We estimate the cost of capital of the conglomerate from the Bloomberg database and use WACC—weighted average cost of capital as a proxy of the average cost of raising capital in the market for a conglomerate. As of October 2012, the values of WACC were equal to 6.8 and 3.4 % for AE and BofA, respectively. We distribute the cost of capital to different BLs based on the portfolio loss rates (following the FED report). By doing so, we characterize different costs of capital that a firm would have needed to raise for each of the BLs. As each of the BLs can be ranked by its level of risk, we assume that the realized loss rates provide a good proxy for the estimation of such costs by the management of the bank. The distribution of costs to various BLs is presented in Table 2. We set negative losses (gains) to zero and the confidence level for VaR at 95 %. Moreover, each BL i is assumed to have equal weight v_i and a deviation e_i equal to 1. The allocated capitals are restricted to non-negative amounts to simplify the analysis.

5.2 Results

We remark that the two objectives are conflicting as described in Sect. 4. The results of different allocation methods are presented in Table 3 for BofA and in Table 4 for AE. For each of the methods presented in Sect. 3 we look at the cost and deviation (note that in the discussion of results "VaR" corresponds to the proportional allocation,

Table 1 List of parameters

Parameter	Definition
c_i	Cost of allocation of capital by BL i
C	Total cost of allocation of capital of the conglomerate
K	Total capital of the conglomerate
K_i	Allocated capital by BL i
v_i	Weight of BL i
e_i	Weight of deviation for BL i

Table 2 Distribution of costs of allocation of capital to different BLs

BL	AE	BofA
First-lien mortgages, domestic (BL1)	0	$0.11C/100$
Junior liens and HELOCs, domestic (BL2)	0	$0.19C/100$
Commercial and industrial (BL3)	$0.36C/100$	$0.10C/100$
Commercial real estate, domestic (BL4)	0	$0.17C/100$
Credit cards (BL5)	$0.46C/100$	$0.32C/100$
Other consumer (BL6)	0	$0.08C/100$
Other loans (BL7)	$0.17C/100$	$0.03C/100$

Table 3 Optimal capital allocations using different methods for BofA

Allocation method	Cost	Deviation	BL1	BL2	BL3	BL4	BL5	BL6	BL7	Total capital
VaR	0.14	174.79	5.91	3.86	3.54	2.18	5.93	1.62	1.05	24.10
CTE	0.14	175.70	5.81	3.84	3.54	2.24	5.82	1.70	1.14	24.10
Covariance	0.12	239.10	3.43	3.89	3.33	3.29	3.47	3.54	3.15	24.10
Dhaene et al. (2012)	0.15	172.69	6.40	3.94	3.57	1.95	6.40	1.26	0.58	24.10
MOMCAP $\alpha = 0$	0.15	172.69	6.40	3.94	3.57	1.95	6.40	1.26	0.58	24.10
MOMCAP $\alpha = 0.5$	0.06	555.25	2.54	0.00	8.80	0.00	0.00	5.69	7.07	24.10
MOMCAP $\alpha = 1$	0.02	3564.10	0.00	0.00	0.00	0.00	0.00	0.00	24.10	24.10

Table 4 Optimal capital allocations using different methods for AE

Allocation method	Cost	Deviation	BL3	BL5	BL7	Total capital
VaR	0.32	16.73	7.28	2.46	5.66	15.40
CTE	0.33	17.51	7.16	2.64	5.60	15.40
Covariance	0.35	48.93	5.28	5.10	5.02	15.40
Dhaene et al. (2012)	0.32	15.60	7.73	1.87	5.80	15.40
MOMCAP $\alpha = 0$	0.32	15.60	7.73	1.87	5.80	15.40
MOMCAP $\alpha = 0.5$	0.24	51.14	4.81	0.00	10.59	15.40
MOMCAP $\alpha = 1$	0.18	146.44	0.00	0.00	15.40	15.40

"Covariance" corresponds to the covariance allocation, "CTE" corresponds to the conditional tail expectation allocation). The results show that in all cases when the cost is not considered, the minimum deviation criterion leads to non-optimal costs. When using our MOMCAP model, the results vary with the chosen level of α. When $\alpha = 0$ the results of our simulations are aligned with the analytical solution given in Eq. 9. With the increasing levels of α, the costs decrease while deviation increases and the capital is allocated to the most cost efficient BLs. Finally, when $\alpha = 1$, the cost is minimized and the total capital is allocated to the least costly BL (BL7—both for BofA and AE). Naturally, this is an extreme case, and managers need to make an informed decision on the chosen level of α by taking into account the complexity of the respective business model and the firm's risk appetite. In practice, the business plans delivered every year by the managers of the corresponding BLs can serve as sources of information used by the decision maker (DM). This information serves as the basis for expressing the DM's preference about the chosen level of α. Since the optimal capital allocation comes at a cost, the deviation is higher when the cost is low (and vice versa, the higher the cost, the lower the deviation). To facilitate the discussion, we depict the efficient frontiers and the results of the simulations in Figs. 1 and 2. These figures show scatterplots of the values of the two objectives given 10,000 feasible solutions generated randomly for the two data sets described in Sect. 5.1. The results suggest that the points lying at the efficient frontier depict the optimum allocation of capital. We see that the solutions of VaR, CTE and Covariance methods are non-optimal since

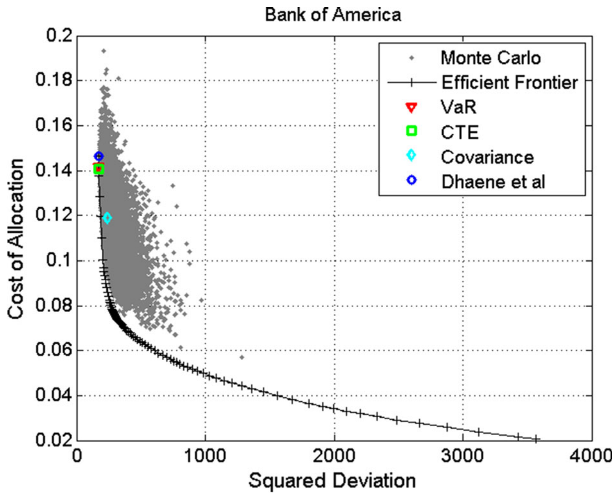

Fig. 1 Efficient frontier for BofA

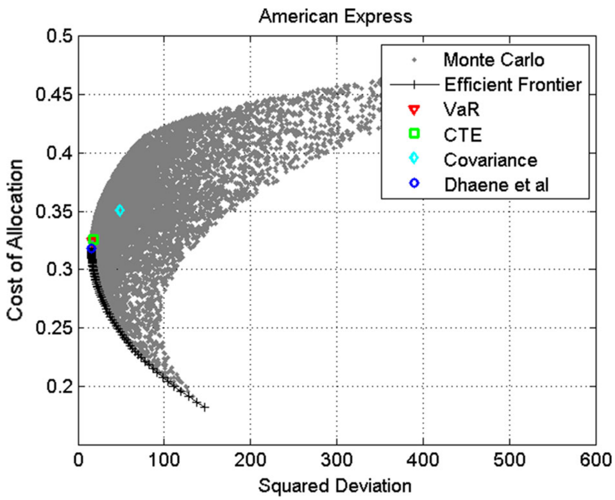

Fig. 2 Efficient frontier for AE

they do not lie along the efficient frontier. In particular, the Covariance method clearly underperforms as compared to other methods. One can also see from these graphs that the smaller values of f_1 do not translate to smaller values of f_2, and vice versa. This behavior can also be observed in the portfolio optimization problem where there are many Pareto optimal solutions.

In Figs. 3 and 4 we show the contribution of each BL to the total allocated capital for the increasing values of α. Interestingly, for higher values of α, the BL with lowest cost has a larger amount of allocated capital. This observation can be attributed to the fact that more weight is given to the cost minimization criterion. As the values of α

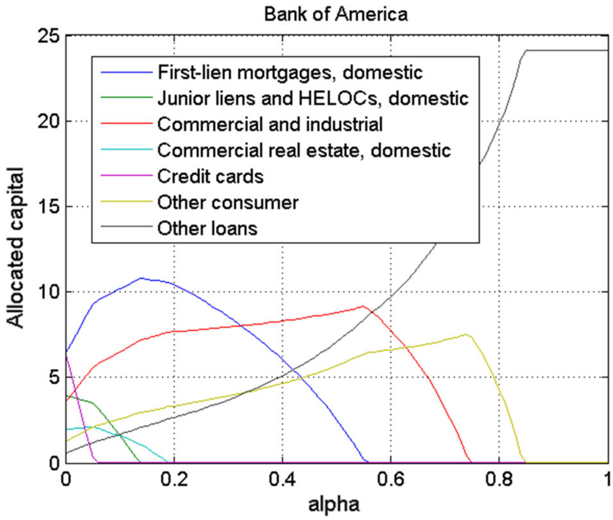

Fig. 3 Capital allocation to different BLs for BoFA

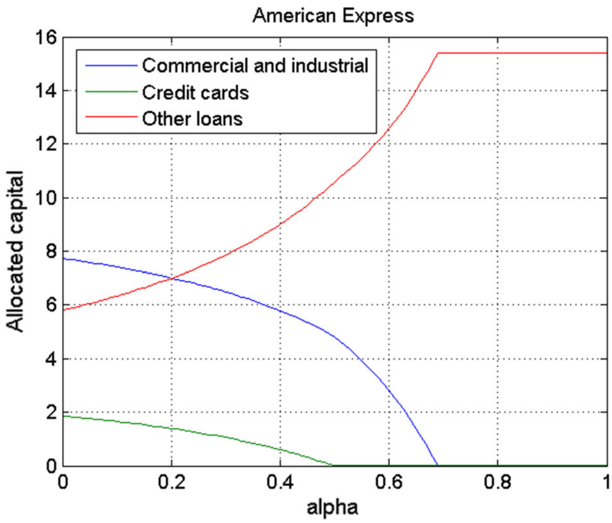

Fig. 4 Capital allocation to different BLs for AE

become lower, the less pronounced is the amount of capital allocated to this BL since the total capital is split across all other BLs.

We have shown through simulation that the optimal solution for the deviation minimization problem is true when we do not consider the costs of allocating capital. In case, when the costs are included in the model, the implications are not so straightforward anymore. We have two conflicting objectives leading to the situation where all analytical methods lead to non-optimal solutions (resulting in a higher deviation when the costs are lower).

6 Conclusions and future research

In this article we extended the optimum capital allocation problem proposed by Dhaene et al. (2012) to a multiobjective optimization problem by incorporating the cost of capital. Our contribution is at least threefold. First, we separate the two optimization objectives in order to decouple the conflicting assumptions of minimizing the cost of capital and the distance of loss and required capital. Second, we solve the two-stage optimization problem by simulating possible scenarios and drawing different maps of the outcomes in the parameter space. We also provide an analytical solution of MOMCAP by using geometric arguments and propose an alternative solution based on Lagrange multipliers. Third, we empirically validate our model through a real-world application of our method. We highlight the differences in optimal capital allocations by comparing the existing capital allocation methods with our mew method for two heterogeneous lending firms based on data from public data sources. Moreover, we develop a theoretical underpinning by linking the optimal capital allocation frontier and the efficient frontier of the Markowitz's modern portfolio theory. In practice, our methodology provides a sophisticated decision making tool which is easy to implement and increases the transparency of risk capital allocation in financial institutions. The use of this model can help the DM to decide on the optimal capital allocation based on the given cost of capital.

However, we are aware of the limitations of this study. Our choice of parameters has only partially been validated with real data. Further research should focus on obtaining and simulating a full distribution of credit losses and incorporating non-trivial correlation structures among the various BLs. Consideration of other risk types would also be an interesting research path. Given the rapidly changing business environment of the international financial industry sector, further development of our multiobjective optimal capital allocation method is thus of interest for academics and practitioners alike.

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